Topology in Physics 2018 - lecture 14

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Today, we will finish the "physics proof" of the Atiyah-Singer index theorem for the Dirac operator D on a curved manifold M. For this operator, as we saw in the previous lecture, the index theorem reads

$$\mathrm{ind}(D) = \int_M \hat{A}(M) \tag{14.1}$$

where the right hand side involves the $\hat{A}$-genus that can be written as

$$\hat{A}(M) = \prod_{j=1}^{n} \frac{x_j/2}{\sinh x_j/2}. \tag{14.2}$$

Here, x_j are the "block diagonalized" components of the anti-symmetric curvature two-form

$$\begin{aligned}
\frac{1}{2\pi} R_{\mu\nu} &\equiv \frac{1}{2\pi}\left(\frac{1}{2} R_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma\right) \\
&= \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & x_n \\ & & & -x_n & 0 \end{pmatrix}.
\end{aligned} \tag{14.3}$$

As before, in the proof below we are closely following Nakahara, section 12.10.

Recall the philosophy of index theorems: we want to relate an **analytical quantity** – the left hand side of (14.1), which is built up of dimensions of kernels – to a **topological quantity**: the right hand side of (14.1), which is the integral of a characteristic class over the manifold. To achieve this, in lecture 12, we have rewritten the left hand side as a path integral with periodic boundary conditions:

$$\mathrm{ind}\,(D) = \int_{\mathrm{PBC}} Dx D\psi \exp\left(-\int_{t=0}^{t=\beta} L[x,\psi] dt\right), \tag{14.4}$$

where, L is the Lagrangian

$$L = \frac{1}{2} g_{\mu\nu}(x)\big(\dot{x}^\mu \dot{x}^\nu + \psi^\mu D_t \psi^\nu\big) \tag{14.5}$$

and D_t is the covariant derivative:

$$D_t\psi^\nu = \dot{\psi}^\nu + \dot{x}^\lambda {\Gamma^\nu}_{\lambda\kappa}(x)\psi^\kappa. \tag{14.6}$$

Our goal for today is to evaluate this path integral and arrive at the right hand side of (14.1).

14.1 β-dependence of the path integral normalization

A very important ingredient in our computation will be the fact that the index, as we discussed in detail in lecture 12, is independent of β. To make use of this fact, let us change $t \to \beta t$ in the path integral (14.4) to obtain

$$\mathrm{ind}\,(D) = \int_{\mathrm{PBC}} DxD\psi \exp\left(-\beta\int_{t=0}^{t=1} L[x,\psi]dt\right). \tag{14.7}$$

The β-dependence is now fully contained in the prefactor of the Lagrangian. Note that this is where in quantum field theories one would write $1/\hbar$, so we can identify β with the inverse of Planck's constant.

This identification is useful if we want to see what it means that this path integral is independent of $\beta \sim 1/\hbar$. In particular, recall that in lecture 2, we obtained a path integral expression for a quantity like $\langle x_N|e^{-H(t_N-t_0)/\hbar}|x_0\rangle$, where $t_N = t_0 + N\delta t$, by repeatedly inserting the identity in the form

$$I = \int dx_n |x_n\rangle\langle x_n| \tag{14.8}$$

at times $t_n \equiv t_i + n\cdot\delta t$ and then evaluating expressions of the form

$$\langle x_{n+1}|e^{-H\delta t/\hbar}|x_n\rangle \tag{14.9}$$

by inserting an additional identity

$$I = \int dp_n |p_n\rangle\langle p_n| \tag{14.10}$$

and evaluating the p-integral in

$$\int dp_n \langle x_{n+1}|p_n\rangle\langle p_n|e^{-H\delta t/\hbar}|x_n\rangle. \tag{14.11}$$

What we swept under the rug in the computation in lecture 2, but what we *are* interested in here, is the normalization of the answer. In particular, since we want to study the $\beta \sim 1/\hbar$-independence of our path integral, we are interested in the factors of $\hbar$ that the above computation results in. (In fact, these factors always appear in the combination $2\pi\hbar$, so we will always use that combination and adjust for aditional numerical factors later.)

Now note that

$$\langle x|p\rangle = Ce^{ipx/\hbar} \tag{14.12}$$

is the position representation of a momentum eigenstate, but what is its normalization C? Clearly, a plane wave cannot be normalized in the canonical way (so that the total probability after integrating over x equals 1), but it can be normalized in the sense that

$$\langle p'|p\rangle = \delta(p - p') \tag{14.13}$$

Inserting a set of x-eigenstates the left hand side becomes

$$\int dx\ \langle p'|x\rangle\langle x|p\rangle = C^2 \int dx\ e^{i(p-p')x/\hbar} \tag{14.14}$$

whereas for the right hand side of (14.13) we can use the Fourier representation of the delta function:

$$\delta(p - p') = \frac{1}{2\pi}\int dy\ e^{i(p-p')y} = \frac{1}{2\pi\hbar}\int dx\ e^{i(p-p')x/\hbar} \tag{14.15}$$

where in the last step we substituted $y = x/\hbar$. Equating the last two expressions, we see that

$$C = \frac{1}{\sqrt{2\pi\hbar}}. \tag{14.16}$$

In (14.11), we therefore obtain two factors of C, but then the integration over p removes one of those, as[1]

$$\int dp\ e^{-\frac{p^2}{2\hbar}} = C^{-1}. \tag{14.17}$$

Thus, alltogether, the quantity (14.11) after doing the p-integrals contains *one* factor of C. For every p-integral, there is one x-integral[2], and so we see that every x_n-itegral in the path integral comes with a factor of C: we should view the $Dx(t)$-integration in the path integral as the limit of a large number of x-integrations normalized as

$$\int Dx(t) = \lim_{N\to\infty}\int \prod \frac{dx_i}{\sqrt{2\pi\hbar}} \tag{14.18}$$

Let us now replace $\beta = 1/\hbar$ and investigate what this normalization means for the β-independence of the index.

[1]Here, we set the mass m appearing in the kinetic energy $p^2/2m$ equal to 1, as we will not be interested in the dependence on this parameter.

[2]At least in the large N limit; in this example there are actually N integrals over p_n and $(N-1)$ integrals over x_n. However, in the situation with periodic boundary conditions that we are interested in, there is one additional integral over $x_0 = x_N$.

14.2 Saddle point approximation to the path integral

Let us begin by looking at a toy model: a single x-integral, which we now normalize according to what we have learned:

$$Z = \sqrt{\frac{\beta}{2\pi}} \int dx \exp\left(-\beta f(x)\right). \tag{14.19}$$

Let us now first assume that the function $f(x)$ has a single extremum at $x = \bar{x}$. We then expand $f(x)$ around this extremum:

$$f(x) = f(\bar{x}) + \frac{1}{2}(x-\bar{x})^2 f''(\bar{x}) + \frac{1}{6}(x-\bar{x})^3 f^{(3)}(\bar{x}) + \dots \tag{14.20}$$

Note that the linear term on the right hand side is absent because we have an extremum at $x = \bar{x}$. Writig $(x - \bar{x}) = y/\sqrt{\beta}$, we can write Z as

$$Z = e^{-\beta f(\bar{x})} \int dy\, e^{-f''(\bar{x})y^2} \, \exp\left(\frac{1}{6}\beta^{-1/2} y^3 f^{(3)}(x_0) + \frac{1}{24}\beta^{-1} y^4 f^{(4)}(x_0) + \dots\right) \tag{14.21}$$

To evaluate the integral, one would normally expand[3]

$$\exp\left(\frac{1}{6}\beta^{-1/2} y^3 f^{(3)}(x_0) + \frac{1}{24}\beta^{-1} y^4 f^{(4)}(x_0) + \dots\right) = \sum_{n=0}^{\infty} a_n(y)\beta^{-n} \tag{14.22}$$

and then evaluate the integral order by order in $\beta^{-1} \sim \hbar$ to get an (asymptotic) $\hbar$-expansion of the result. However, we are interested in the situation where our final answer is $\hbar$-*independent*, and so we see that we only need the leading term $a_0 = 1$ in the expansion. Moreover, the result can only be $\hbar$-idependent if $f(\overline{x}) = 0$ so that the prefactor does not contribute. Thus, in the β-independent case, we really only need to evaluate the "quadratic fluctuation part" of the integral around the saddle point:

$$Z_{quadratic} = \int dy\, e^{-f''(\bar{x})y^2} \tag{14.23}$$

This makes our lives a lot easier: evaluating a β-independent path integral simply boils down to Gaussian integration! Of course, this is not quite as simple as the above example may appear to indicate, as we still need to address two issues:

1. There may be (and actually: will be) more than a single saddle point for our action,
2. We must know how to do *infinte-dimensional* Gaussian integrals.

We will now address these issues one by one.

[3] Note that odd powers of $\beta^{-1/2}$ appear with odd powers of y, and will therefore not contribute to the integral

14.3 Saddle points of the supersymmetric action

Recall that the quantity in the exponent of our path integral is the supersymmetric action

$$S = \int_0^1 dt\ \frac{1}{2} g_{\mu\nu}(x) \big(\dot{x}^\mu \dot{x}^\nu + \psi^\mu D_t \psi^\nu \big) \tag{14.24}$$

The saddle points of the path integral are the configurations $x^\mu(t)$m $\psi^\mu(t)$ for which the action does not change to first order – that is: the solutions to the **Euler Lagrange equations** of motion! The derivation of those equations is tedious but straightforward (see Nakahara); here, we simply state the result:

$$\begin{aligned} 0 &= D_t \psi^\mu \\ 0 &= -g_{\lambda\mu} D_t \dot{x}^\mu + \frac{1}{2} R_{\mu\nu\lambda\rho} \psi^\mu \psi^\nu \dot{x}^\rho. \end{aligned} \tag{14.25}$$

One set of solutions is clear from these equations: the *constant* solutions

$$x^\mu(t) = x_0^\mu, \qquad \psi^\mu(t) = \psi_0^\mu \tag{14.26}$$

Note that for these solutions, one moreover has that $S = 0$, so that indeed their zeroeth order contribution vanishes. In fact, it can be shown that these are the *only* saddle points that contribute in the limit $\beta \to 0$, and since our result is β-independent, we therefore only need to take those into account.

So our saddle points are simple: they are the *constant* configurations in time. (These clearly also satisfy the periodic boundary condition.) Now what do we do with the fact that there is more than a single saddle point? The answer is: we need to **add the contributions of the different saddle points**. This may not seem trivial from what we have said so far, and in fact in our simple example in the previous subsection it is far from obvious that if there are two or more saddle points for $f(x)$, one should add their perturbative contributions. In fact there is a long story in that case: the separate contributions from saddle points are only defined as formal power series in $\hbar$ as they are *asymptotic* and do not converge for any value of $\hbar$. To turn them into actual functions one needs to Borel resum one of these power series, and in the process it becomes clear that the contributions of the Borel sums of the power series around other saddle points must be added as well. This leads into the topic of *resurgence*, a very interesting topic but not one we will delve into here.

For our path integral, we are in a rather different situation, as there are *no* $\hbar$-corrections, and so the power series are not asymptotic but actually terminate. So how do we see that we need to add contributions from all different saddle points here? The easiest (though admittedly handwaving) way to see this is to Fourier expand the periodic paths:

$$\begin{aligned} x^\mu(t) &= x_0^\mu + \xi^\mu(t) \\ &= x_0^\mu + \sum_{n\neq 0} \xi_n^\mu e^{2\pi i n t}. \end{aligned} \tag{14.27}$$

Here, we have set $\beta = 1$ as we already discussed how to get the required, β-independent result. Now we "change variables" in the path integral, and integrate over the modes x_0^μ and ξ_n^μ instead of over the individual values $x^\mu(t)$. Clearly, in this way one also integrates over all periodic paths. One may of course worry about a Jacobian coming from this change of variables, but since (a) the path integral normalization is not well-defined to begin with, (b) we have already made sure that the final answer has the correct β-(in)dependence and (c) we will fix the numerical normalization of our result in an independent way later on, we will put on our "physicist's hat" and will not worry about this Jacobian.

The nice thing about the above change of integration variables is that now it is clear that we must integrate over all constant modes x_0^μ, so that indeed we "add" all contributions from the different saddle points. All that remains is to evaluate the contributions coming from the terms quadratic in the fluctuations ξ_n^μ.

Before evaluating those, let us note (again, without detailed proof) that we can do a similar change of variables for the fermionic coordinates: one can write

$$\psi^\mu(t) = \psi_0^\mu + \sum_{n\neq 0} \eta_n^\mu e^{2\pi i n t} \tag{14.28}$$

and integrate over the Grassmann variables ψ_0^μ and η_n^μ. In fact, the reader may have wondered what the "equation of motion" $D_t\psi^\mu = 0$ actually meant, as $\psi^\mu(t)$ cannot "take on any values", and so cannot be "constant" either. It should now be clear what is meant: the saddle points ("solutions to the Euler-Lagrange equations") are parameterized by the modes ψ_0^μ, and the fluctuations around them by the modes η_n^μ.

14.4 Gaussian integrals and determinants

As we learned from our simple example, we now need to evaluate the quadratic fluctuations around our saddle points. To this end, it is very convenient to choose a particular bosonic saddle point x_0 and choose coordinates such that the metric at x_0 satisfies

$$g_{\mu\nu}(x_0) = \delta_{\mu\nu}, \qquad \frac{\partial}{\partial x^\lambda} g_{\mu\nu}(x_0) = 0. \tag{14.29}$$

Of course, we cannot impose any further conditions on higher derivatives of the metric, as those are determined by the curvature of the manifold. In these coordinates, we can now expand the action

$$S = \int_0^1 dt\, \frac{1}{2} g_{\mu\nu}(x)\big(\dot{x}^\mu \dot{x}^\nu + \psi^\mu D_t \psi^\nu\big) \tag{14.30}$$

up to second order in the fluctuations, and obtain after a short computation that

$$S_2 = \int_0^1 dt\, \left(\frac{1}{2}\delta_{\mu\nu}\dot{\xi}^\mu\dot{\xi}^\nu + \frac{1}{2}\delta_{\mu\nu}\eta^\mu\dot{\eta}^\nu + \frac{1}{2}\tilde{R}_{\mu\nu}(x_0)\xi^\mu\dot{\xi}^\nu\right) \tag{14.31}$$

where

$$\tilde{R}_{\mu\nu}(x_0) = \frac{1}{2} R_{\mu\nu\rho\sigma}(x_0)\psi_0^\rho \psi_0^\sigma \tag{14.32}$$

is very similar to the curvatur two-form, but with dx^μ replaced by the (equally anti-commuting) Grassmann variable ψ_0^μ.

Let us do one further partial integration, and rewrite the quadratic part of the action as

$$S_2 = \int_0^1 dt \; \frac{1}{2}\xi^\mu \left(-\delta_{\mu\nu}\frac{d^2}{dt^2} + \tilde{R}_{\mu\nu}(x_0)\frac{d}{dt} \right) \xi^\nu + \frac{1}{2}\eta^\mu \left(\delta_{\mu\nu}\frac{d}{dt} \right) \eta^\nu \tag{14.33}$$

Now, we are ready to do the Gaussian integrals. Recall that for a *bosonic* Gaussian integral over several variables, we have

$$\int d^n y \exp\left(-\frac{1}{2} y^i A_{ij} y^j \right) = \frac{2\pi^{n/2}}{\sqrt{\det A}}. \tag{14.34}$$

For a *fermionic* Gaussian integral (i.e. one over Grassmann variables), we have seen in exercise 9.1 that

$$\int d^n \theta \exp\left(-\frac{1}{2} \theta^i B_{ij} \theta^j \right) = \sqrt{\det B}. \tag{14.35}$$

As a result, we can write

$$\int D\xi D\eta \, e^{-S_2} = \mathcal{N} \sqrt{\frac{\det' \left(\delta_{\mu\nu}\frac{d}{dt} \right)}{\det' \left(-\delta_{\mu\nu}\frac{d^2}{dt^2} + \tilde{R}_{\mu\nu}(x_0)\frac{d}{dt} \right)}}. \tag{14.36}$$

where the prime is inserted to remind us that both ξ and η do *not* include the zero modes. In this expression, $\mathcal{N}$ is a normalization factor that we still need to determine; it should be built up from all of the factors of $\sqrt{2\pi}$ that have appeared in our expressions, but since we have an infinite number of those from (14.34) in the numerator and an infinite number from (14.18) in the denominator, we of course need to be careful in simply crossing them out. The easiest way to determine $\mathcal{N}$ is to evaluate all of the expressions on a *flat* manifold of dimension $2d$; this is done in Nakahara, and it turns out the correct normalization is

$$\mathcal{N} = i^d. \tag{14.37}$$

We plug this in in (14.36), and also note that since determinants of products factorize[4], we can divide out the common factor of $\delta_{\mu\nu}\frac{d}{dt}$ in numerator and denominator. This leads to

$$\int D\xi D\eta \, e^{-S_2} = i^d \det' \left(-\delta_{\mu\nu}\frac{d}{dt} + \tilde{R}_{\mu\nu}(x_0) \right)^{-1/2}. \tag{14.38}$$

[4]This is in fact something that needs to be shown for properly defined infinite-dimensional determinants, but it turns out to be true if e.g. the zeta-function regularization is used. Also, note that in one case we are taking a determinant over bosonic modes and in the other case over fermionic modes, but since we are dealing with a supersymmetric theory and zero modes are excluded in this computation, these can indeed be mapped to each other in a 1-to-1 fashion.

where the determinant is over all bosonic non-zero modes. Let us focus on the modes ξ_n^μ for a single value of n first. For these modes, after block-diagonalizing $\mathcal{R}_{\mu\nu}(x_0)$, we can write the determinant as

$$
\begin{aligned}
& \det \begin{pmatrix} -\frac{d}{dt} & y_1 & & & \\ -y_1 & -\frac{d}{dt} & & & \\ & & \ddots & & \\ & & & -\frac{d}{dt} & x_n \\ & & & -x_n & -\frac{d}{dt} \end{pmatrix} \\
&= \det \begin{pmatrix} -2\pi i n & y_1 & & & \\ -y_1 & -2\pi i n & & & \\ & & \ddots & & \\ & & & -2\pi i n & x_n \\ & & & -x_n & -2\pi i n \end{pmatrix} \\
&= \prod_{i=1}^{d} \left(y_i^2 - (2\pi n)^2\right) \qquad (14.39)
\end{aligned}
$$

To obtain the full determinant, we need to multiply this result for all $n \neq 0$. Of course, positive and negative values of n give the same result, so we obtain

$$
\begin{aligned}
\det{}' \left(-\delta_{\mu\nu}\frac{d}{dt} + \tilde{R}_{\mu\nu}(x_0)\right) &= \prod_{i=1}^{d}\prod_{n\geq 1}\left(y_i^2 - (2\pi n)^2\right)^2 \\
&= \left[\prod_{i=1}^{d}\prod_{n\geq 1}(2\pi n)^2 \prod_{n\geq 1}\left(1 - \frac{y_i^2}{(2\pi n)^2}\right)\right]^2 \qquad (14.40)
\end{aligned}
$$

The second factor in this expression is Euler's product formula for the sine:

$$
\prod_{n\geq 1}\left(1 - \frac{y^2}{(2\pi n)^2}\right) = \frac{\sin(y/2)}{y/2} \qquad (14.41)
$$

If you have never seen this formula before, we will prove it in exercise 2. The first factor in the determinant seems more worrisome, as it is essentially the product of all positive integers, which appears to wildly diverge. Fortunately, there is a method to regularize such products leading to finite results; this method of ζ-*function regularization* is presented in exercise 1. The result is surprisingly simple:

$$
\prod_{n\geq 1}(2\pi n)^2 \to 1. \qquad (14.42)
$$

Thus, we find that

$$
\det{}' \left(-\delta_{\mu\nu}\frac{d}{dt} + \tilde{R}_{\mu\nu}(x_0)\right) = \prod_{i=1}^{d}\left(\frac{\sin(y_i/2)}{y_i/2}\right)^2 \qquad (14.43)
$$

14.5 The index theorem

Now, we can gather all ingredients that we have calculated. The formula of our index then becomes

$$\mathrm{ind}(D) = \int \prod_{\mu=1}^{2d} dx_0^\mu \prod_{\mu=1}^{2d} d\psi_0^\mu \; i^d \prod_{i=1}^{d} \left(\frac{y_1/2}{\sin(y_1/2)} \right) \tag{14.44}$$

Here, the integral over the bosonic zero modes clearly is an integral over the manifold M. What about the integral over the fermionic zero modes? Recall that integration over Grassmann variables equals differentiation, and so this integration picks out the term in the y-expansion in which every ψ_0^μ appears once. In fact, if we now replace $\psi_0^\mu \to dx^\mu$, this is precisely what the notation in the index theorem implies. The above formula therefore simplifies in form to

$$\mathrm{ind}(D) = \int_M i^d \prod_{i=1}^{d} \left(\frac{x_i/2}{\sin(x_i/2)} \right). \tag{14.45}$$

where we indicated the replacement $\psi_0^\mu \to dx^\mu$ by replacing y_i with x_i. The very last step in our computation is to note that $\frac{x}{\sin x}$ is an *even* function in x, so that the result is only nonzero if d is even. In that case, $i^d = (-1)^{d/2}$, and so we can obtain the same result by removing the factors of i and replacing the sine by a sinh:

$$\mathrm{ind}(D) = \int_M \prod_{i=1}^{d} \left(\frac{x_i/2}{\sinh(x_i/2)} \right). \tag{14.46}$$

The right hand side now contains exactly the $\hat{A}$-genus:

$$\mathrm{ind}(D) = \int_M \hat{A}(M). \tag{14.47}$$

Thus, we have completed the physics proof of the Atiyah-Singer index theorem for the Dirac operator.